$\sqrt{\Delta} \to \Delta \geq 0$

$\ln(\Delta) \to \Delta > 0$

Section 14.1 Functions of Several Variables

A function with two inputs $f(x, y)$ is called a function of two variables. Examples include:

$$f(x,y) = \sqrt{y - x^2}, \qquad f(x,y) = \frac{1}{xy}, \qquad V(r,h) = \pi r^2 h$$

The **domain** of a function of two variables $f(x, y)$ is define to be the set of points in the xy-plane for which the function generates real numbers. The **range** of a function of two variables $f(x, y)$ is the set of all possible output values and is a subset of the real line $\mathbb{R}$.

Functions of Three Variables: A function with three inputs $f(x, y, z)$ is called a function of three variables. Examples include:

$$f(x,y,z) = \frac{1}{1 + x^2 + y^2 + z^2}, \qquad f(x,y,z) = xy \ln z$$

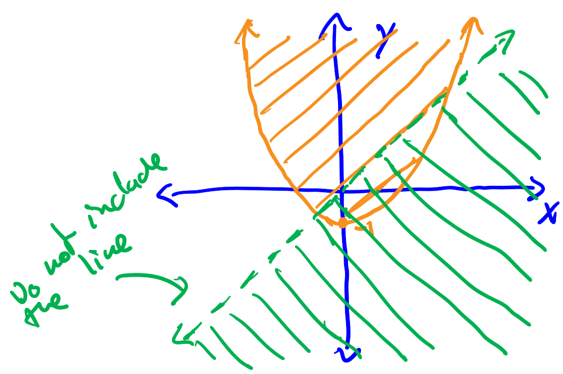
The **domain** of a function of three variables $f(x, y, z)$ is define to be the set of points in the xyz-space for which the function generates real numbers. The **range** of a function of three variables $f(x, y, z)$ is the set of all possible output values and is a subset of the real line $\mathbb{R}$.

Ex1. Find and sketch the domain of $f(x, y) = \sqrt{1 + y - x^2}\,\ln(x - y)$.

Belongs to

Domain: $D = \{ (x,y) \in \mathbb{R}^2 \mid 1 + y - x^2 \geq 0 \text{ and } x - y > 0 \}$

Such that (may also be ":")

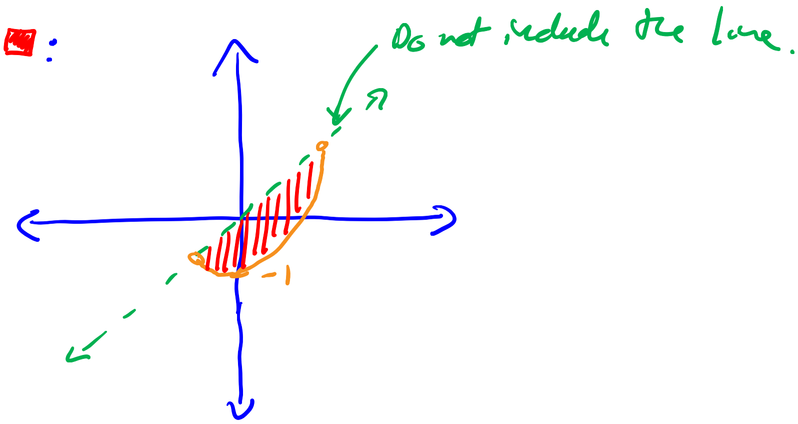


$* \ 1 + y \geq x^2 \Rightarrow y \geq x^2 - 1$

$* \ x + y > 0 \Rightarrow x > y$

(Test pt: $(0, -1)$
$0 > -1$ ✓)

Then, sketch the domain:

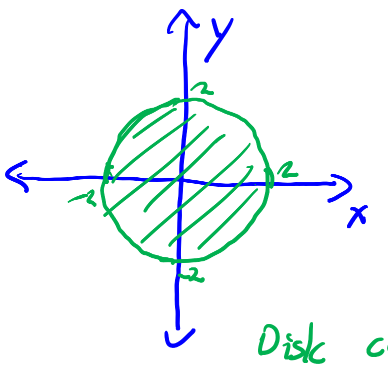


Ex2. Find and sketch the domain. Find the range.

a) $f(x, y) = \sqrt{4 - x^2 - y^2}$.

Domain: $\{(x, y) \in \mathbb{R}^2 \mid 4 - x^2 - y^2 \geq 0\}$

$x^2 + y^2 \leq 4$



Disk centered at $(0,0)$ and radius 2. (Include the boundary).

Range:

$x^2 + y^2 \geq 0$

$-x^2 - y^2 \leq 0$ add 4

$0 \leq 4 - x^2 - y^2 \leq 4$

$0 \leq \sqrt{4 - x^2 - y^2} \leq 2$

so, range is $[0, 2]$

b) $f(x, y, z) = \dfrac{3}{1 + x^2 + y^2 + z^2}$

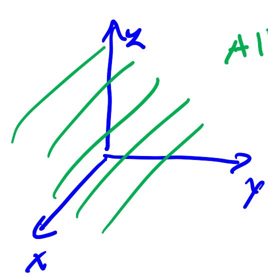
In this example, the denominator is NEVER zero: $x^2 + y^2 + z^2 \geq 0$

$1 + x^2 + y^2 + z^2 \geq 1 > 0$

Domain: $\{(x, y, z) \in \mathbb{R}^3 \mid x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}\}$

Domain in $\mathbb{R}^3$:

All points in $\mathbb{R}^3$



Range $x^2 + y^2 + z^2 \geq 0$

$1 + x^2 + y^2 + z^2 \geq 1$

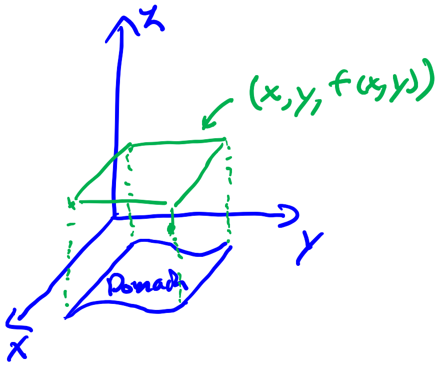
$1 \geq \dfrac{1}{1 + x^2 + y^2 + z^2}$

$3 \geq \dfrac{3}{1 + x^2 + y^2 + z^2} > 0$

so the range is $(0, 3]$

Graphs and Level Curves

The set of points $(x, y, f(x, y))$ in $\mathbb{R}^3$, for (x, y) in the domain of f, is called the graph of f. The graph of f is a surface and is also denoted by $z = f(x, y)$.



Ex3. Sketch the graph of the function $\underbrace{f(x, y)}_{z} = 1 + \sqrt{4 - x^2 - y^2}$.

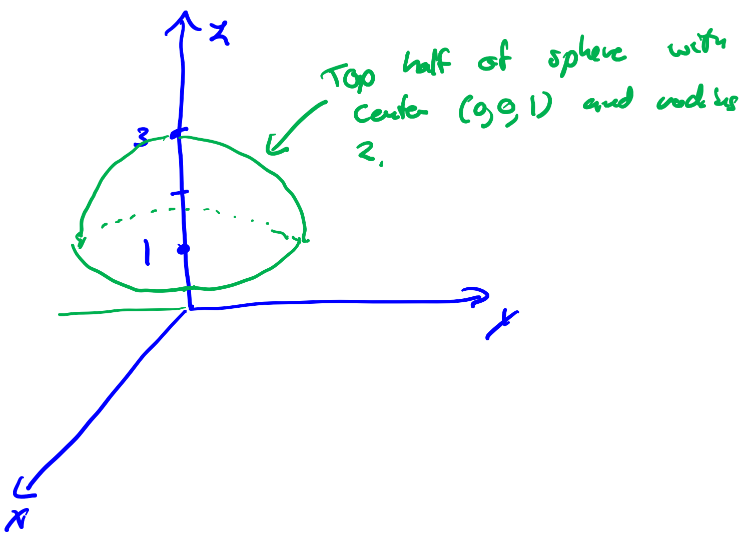
Range: [1, 3]

$z = 1 + \sqrt{4 - x^2 - y^2}$

$z - 1 = \sqrt{4 - x^2 - y^2}$

$(z-1)^2 = 4 - x^2 - y^2$

$x^2 + y^2 + (z-1)^2 = 4$



Level Curves: How to visualize a function $z = f(x, y)$?

- Plotting the graph of the surface $z = f(x, y)$ by using traces.
- By **contour map**, that is, plotting in the xy-plane curves of the form

$$f(x, y) = c \quad (c \text{ is a fixed constant})$$

  i.e., we slice the graph by horizontal planes $z = c$. The set of points in the xy-plane where $f(x, y)$ has a constant value $f(x, y) = c$ is called level curve of f.

Ex4. Let $f(x, y) = 7 - x^2 - y^2$. Sketch the level curves $f(x, y) = c$ where $c = -2, 0, 3, 7, 9$. Then sketch the surface $z = f(x, y)$.

$c=-2$: $7-x^2-y^2=-2 \Rightarrow x^2+y^2=9$

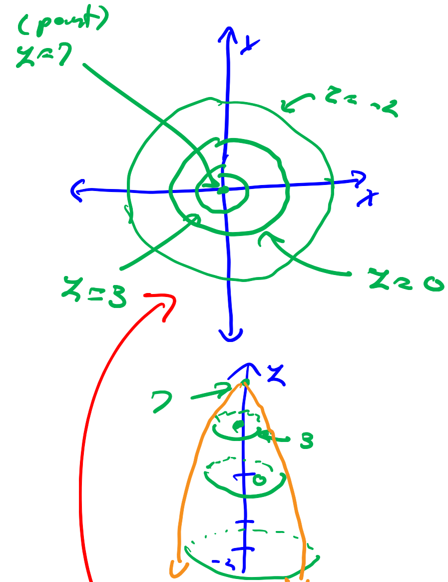
$c=0$: $7-x^2-y^2=0 \Rightarrow x^2+y^2=7$

$c=3$: $7-x^2-y^2=3 \Rightarrow x^2+y^2=4$

$c=7$: $7-x^2-y^2=7 \Rightarrow x^2+y^2=0$

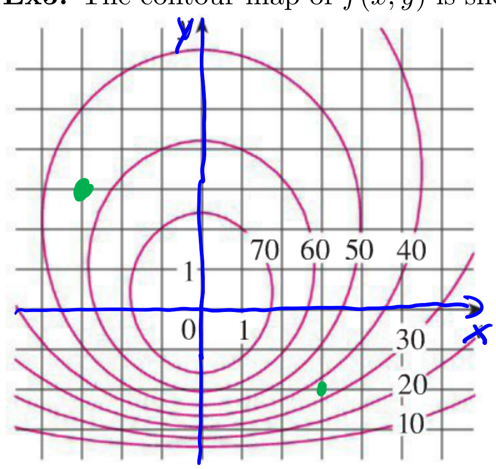
$c=9$: $7-x^2-y^2=9 \Rightarrow x^2+y^2=-2$ "nothing"

$z=f(x,y)$
$z=7-x^2-y^2$
$z-7=-x^2-y^2$



This graph is called a contour plot or contour map

Ex5. The contour map of $f(x, y)$ is shown. Use it to estimate the values of $f(-3, 3)$ and $f(3, -2)$.



$f(-3,3) \approx 56$ ← Between 50 and 60 lines

$f(3,-2) \approx 35$ ← Between 30 and 40 lines

Section 14.2: Limits and Continuity

A function $f(x,y)$ approaches to the *limit* L as (x,y) approaches (a,b), written

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if the distance between $f(x,y)$ and L becomes arbitrarily small whenever the distance from (x,y) to (a,b) is made sufficiently small (but not 0).

Caution: The point (a,b) does not need to be in the domain of $f(x,y)$. For example

$$\lim_{(x,y)\to(1,1)} \overset{f(x,y)}{\boxed{\frac{x-y}{x^2-y^2}}} = \lim_{(x,y)\to(1,1)} \frac{x-y}{(x+y)(x-y)} = \lim_{(x,y)\to(1,1)} \frac{1}{x+y} = \frac{1}{2}$$

Limit Notation

exists even though $(1,1)$ is not in the domain!

As with single-variable functions, the limit of the sum of two functions is the sum of their limits (**when they both exist**), with similar results for the limits of the differences, constant multiples, quotients, powers, and roots.

Theorem Let L, M, and k be real numbers and assume that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L, \qquad \lim_{(x,y)\to(a,b)} g(x,y) = M$$

Then the following rules hold.

i. Sum rule: $$\lim_{(x,y)\to(a,b)} \Big\{f(x,y)+g(x,y)\Big\} = L+M$$

ii. Difference rule: $$\lim_{(x,y)\to(a,b)} \Big\{f(x,y)-g(x,y)\Big\} = L-M$$

iii. Constant multiple rule: $$\lim_{(x,y)\to(a,b)} \Big\{k\cdot f(x,y)\Big\} = k\cdot L$$

iv. Product rule: $$\lim_{(x,y)\to(a,b)} \Big\{f(x,y)\cdot g(x,y)\Big\} = L\cdot M$$

v. Quotient rule: $$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M} \quad \text{if} \quad M\neq 0$$

vi. Power rule: $$\lim_{(x,y)\to(a,b)} \Big\{f(x,y)\Big\}^p = L^p \quad \text{if} \quad p>0 \text{ and } f(x,y)>0.$$

Ex1. Compute $\lim\limits_{(x,y)\to(5,2)} \dfrac{x+2y-9}{\sqrt{x+2y}-3}$.

Note $(5,2)$ is not in the domain of f.

Let $f(x,y) = \dfrac{x+2y-9}{\sqrt{x+2y}-3}$

$$\lim_{(x,y)\to(5,2)} \frac{(x+2y-9)}{(\sqrt{x+2y}-3)} \cdot \frac{(\sqrt{x+2y}+3)}{(\sqrt{x+2y}+3)} = \lim_{(x,y)\to(5,2)} \frac{\cancel{(x+2y-9)}(\sqrt{x+2y}+3)}{\cancel{(x+2y-9)}}$$

$$= \lim_{(x,y)\to(5,2)} (\sqrt{x+2y}+3)$$

$$= \sqrt{5+2(2)}+3 = 6.$$

Ex2. Compute $\lim\limits_{(x,y)\to(1,1)} \left\{\dfrac{x^3-y^3}{x-y} - \dfrac{x-y}{x^2-y^2}\right\}$.

$(1,1)$ is Not in the domain

$$= \lim_{(x,y)\to(1,1)} \left\{\frac{(x-y)(x^2+xy+y^2)}{x-y} - \frac{x-y}{(x-y)(x+y)}\right\}$$

$$= \lim_{(x,y)\to(1,1)} \left\{x^2+xy+y^2 - \frac{1}{x+y}\right\}$$

$$= (1)^2+(1)(1)+(1)^2 - \frac{1}{1+1} = 3 - \frac{1}{2} = \frac{5}{2}.$$

$\frac{0}{0}$ (Indeterminate Form)

Find

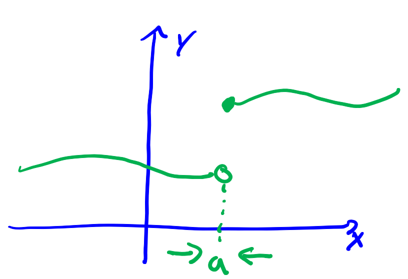
a) $\lim\limits_{x\to 0} \dfrac{x^2}{x} = \lim\limits_{x\to 0} x = 0$

b) $\lim\limits_{y\to 0} \dfrac{y}{2y} = \lim\limits_{y\to 0} \dfrac{1}{2} = 1/2$

c) $\lim\limits_{y\to 0^+} \dfrac{y}{5y^2} = \lim\limits_{y\to 0^+} \dfrac{1}{5y} = +\infty$

(Does not exist)

∞ is not a number



Showing a Limit Does Not Exist

In order for

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

the function $f(x,y)$ must approach the number L regardless of the path along which (x,y) approaches (a,b).

Two Path Test:
If a function $f(x,y)$ has different limits along two different paths as (x,y) approaches (a,b), then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

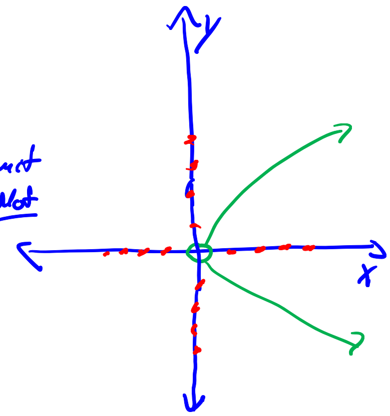
Ex3. Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2x}{x^2+y^4}$ exist? Explain

$f(x,y)=\dfrac{y^2x}{x^2+y^4}$

Note: $(0,0)$ is not in the domain.

Path	$f(x,y)$	limit
$y=0$	$f(x,y)=\frac{0^2x}{x^2+0^4}=\frac{0}{x}=0$	$\lim_{x\to0} f(x,0)=\lim_{x\to0} 0=\boxed{0}$
$x=0$	$f(0,y)=\frac{y^2(0)}{0^2+y^4}=\frac{0}{y^4}=0$	$\lim_{y\to0} f(0,y)=\lim_{y\to0} 0=\boxed{0}$
$x=y^2$	$f(y^2,y)=\frac{y^2(y^2)}{(y^2)^2+y^4}=\frac{y^4}{2y^4}=\frac{1}{2}$	$\lim_{y\to0} f(y^2,y)=\lim_{y\to0}\frac{1}{2}=\boxed{\frac{1}{2}}$

$\neq$ the limit does Not exist



Ex4. Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^3}{xy}$ exist? Explain

$f(x,y)=\dfrac{x^2+y^3}{xy}$

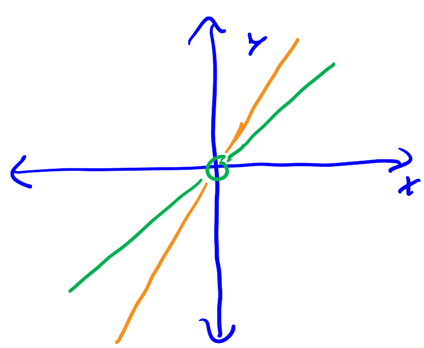
Note: $(0,0)$ is not in the domain

Note: we cannot use the paths $x=0$, $y=0$ bc we cannot divide by zero.

Path	$f(x,y)$	limit
$x=y$	$f(x,x)=\frac{x^2+x^3}{x(x)}=1+x$	$\lim_{x\to0} f(x,x)=\lim_{x\to0}(1+x)=\boxed{1}$
$y=2x$	$f(x,2x)=\frac{x^2+8x^3}{x(2x)}=\frac{x^2(1+8x)}{2x^2}=\frac{1+8x}{2}$	$\lim_{x\to0} f(x,2x)=\lim_{x\to0}\frac{1+8x}{2}=\boxed{\frac{1}{2}}$

$\neq$

DNE

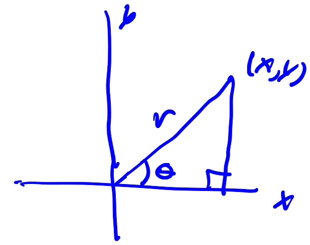


good paths to try:

$x=0$ $\quad$ $y=mx,\ m\neq0$

$y=0$ $\quad$ $y=x^2$

$x=y$ $\quad$ $x=y^2$



Evaluating Limits with Polar Coordinates: The particular case

$$\lim_{(x,y)\to(0,0)} f(x,y).$$

Suppose you try different paths(such as all lines through the origin, $y = mx$, parabolas, etc) and you notice you keep getting the same number. This leads to believe the limit exists. One way to analyze the existence of this limit is by using polar coordinates $\boxed{x = r\cos\theta,\ y = r\sin\theta}$ and taking the limit as $r \to 0^+$.

$r^2 = x^2+y^2$

- If we can show that there are two functions $g(r)$ and $h(r)$ such that

$$g(r) \le f(r\cos\theta, r\sin\theta) \le h(r)$$

for all $\theta \in [0, 2\pi]$ and that $\lim_{r\to 0^+} g(r) = \lim_{r\to 0^+} h(r) = L$, then we have that

$$\lim_{r\to 0^+} f(r\cos\theta, r\sin\theta) = L$$

by the squeeze theorem. Under these settings, the limit in cartesian coordinates does exist and equals L.

Ex5. Use polar coordinates to analyze the following limits:

a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2+2y^2+x^4}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{2(x^2+y^2)+x^4}{(x^2+y^2)} = \lim_{(x,y)\to(0,0)} \left(2 + \frac{x^4}{x^2+y^2}\right)$

Now, $\displaystyle\lim_{r\to 0^+}\left(2 + \frac{r^4\cos^4\theta}{r^2}\right) = \lim_{r\to 0^+}\left(2 + r^2 + \cos^4\theta\right)$

We have

$0 \le \cos^4\theta \le 1$

$0\cdot r^2 \le r^2\cos^4\theta \le 1(r^2)$

$\boxed{2} \le 2 + r^2\cos^4\theta \le \boxed{r^2+2}$ then, $\lim_{r\to 0^+} 2 = 2$, and $\lim_{r\to 0^-} (r^2+2) = 2$

By squeeze thm, $\lim_{r\to 0^+} (2 + r^2 + \cos^4\theta) = 2$

so $\displaystyle\lim_{(x,y)\to(0,0)} \left(\frac{2x^2+2y^2+x^4}{x^2+y^2}\right) = 2.$

b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(\sqrt{x^2+y^2}\,)}{\sqrt{x^2+y^2}}$

$\displaystyle\lim_{r\to 0^+} \frac{\sin(\sqrt{r^2})}{\sqrt{r^2}} = \lim_{r\to 0^+} \frac{\sin(r)}{r} = \lim_{r\to 0^+} \frac{\cos(r)}{1} = \boxed{1}$ (L'Hopital's Rule)

Continuity: A function of two variables $f(x,y)$ is continuous at the point (a,b) if:

(1) f is defined at (a,b)

(2) $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists.

(3) $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Note: The sums, differences, products, and quotients (as long as denominator is not zero) of continuous functions are continuous on their domains.

Ex6. For which value of the constant k is the function f continuous at the origin?

$$f(x,y) = \begin{cases} \dfrac{3x^2y - x^2 - y^2}{x^2+y^2} \quad , & (x,y) \neq (0,0). \\ \\ 2k \quad , & (x,y) = (0,0). \end{cases}$$

$-|3| \le 3 \le |3|$

$-|-5| \le -5 \le |-5|$

$-|y| \le y \le |y|$

① f is defined at $(a,b) = (0,0)$: $f(0,0) = 2k$

② $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \left\{ \dfrac{3x^2y - x^2 - y^2}{x^2+y^2} \right\} = \lim\limits_{(x,y)\to(0,0)} \left\{ \dfrac{3x^2y}{x^2+y^2} - 1 \right\}$

Note: $x^2 \le x^2+y^2 \Rightarrow 0 \le \dfrac{x^2}{x^2+y^2} \le 1$ AND $-|y| \le y \le |y|$

$$-|y| \le \frac{-|y|\,x^2}{x^2+y^2} \le \frac{x^2y}{x^2+y^2} \le |y|\frac{x^2}{x^2+y^2} \le |y|$$

Now $\lim\limits_{(x,y)\to(0,0)} -|y| = 0$, and $\lim\limits_{(x,y)\to(0,0)} |y| = 0$

so $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2+y^2} = 0$

then $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 3(0) - 1 = -1$ (limit exists)

③ we want $\underbrace{\lim\limits_{(x,y)\to(0,0)} f(x,y)}_{-1} = \underbrace{f(0,0)}_{2k}$

so $\boxed{k = -1/2}$.

TO DO:

(1) Does $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{y}$ exist? Explain

(2) Show that $f(x,y) = \dfrac{3x^2y}{x^4+y^2}$ has no limit as (x,y) approaches $(0,0)$.